Quantum Superposition and Entanglement

EE599-001 & EE699-010, Spring 2026

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Physics vs. Comp. Eng.

- Some terms don't mean the same thing…
  - **Qubit**: mathematical vector in Hilbert space, realized by quantum properties
  - **State**: mathematical description of current condition of a time-evolving system
  - **Gate**: unitary transform of a wave function to evolve state over time
  - **Quantum Circuit**: a sequence of gates
  - **Adiabatic**: implies cyclic relaxation, not just thermodynamic reversibility

Quantum Wave Function: ψ

- **ψ(x)** or **ψ** … aka, **Psi**
  - Wave shape; solves Schrödinger equation
  - A wave function with a complex value, a **probability density function** (**PDF**)
  - $|\psi|^2$ is probablity of a measurement outcome, normalized so sum over all possible is 1
- If $|\psi_1>$ and $|\psi_2>$ are potential solutions, then $a|\psi_1> + b|\psi_2>$ is also a solution

Superposition

- Suppose both **|live_cat>** and **|dead_cat>** are valid solutions
- Then √½(**|live_cat>** + **|dead_cat>**) is a **superposition** in which they are equiprobable
- Does this mean a cat can be simultaneously live and dead? *This is math...*
- An equiprobable superposition of 1 and 0 is: **√½**(**|1>** + **|0>**)

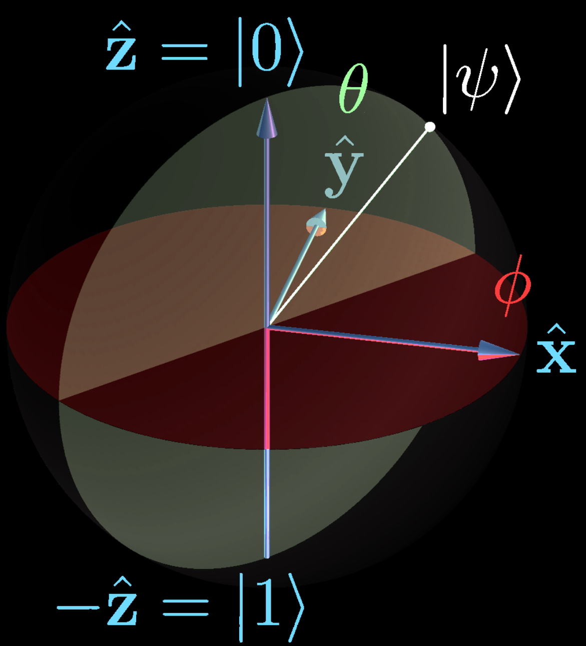
Interference

- Measurement gets you one value
- Wave functions can interfere
- Interference is largely about phase...
- We'll discuss this more later, but interference allows sampling the wave function in a way that acts more like summing than selecting

Decoherence

- Quantum wave functions are fragile
  - Any stray energy (noise) can interact
  - Phase information is particularly fragile (phase error >> than wave function collapse)
- Main implications:
  - A superposed state does not last forever
  - Each operations adds noise & imprecision
  - Quantum computation inherently unreliable; error correction methods are needed

Bloch Sphere Qubit Model



$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

$$= \cos(\theta/2)|0\rangle + (\cos\phi + i\sin\phi)\sin(\theta/2)|1\rangle$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$

- Visualizes wave function of a single qubit
- Probability by coordinates on sphere surface

Entanglement

- An equiprobable superposition of 1 and 0 is:
  $\sqrt{\tfrac{1}{2}}(|1> + |0>)$
- Suppose I have two such superpositions in two separate qubits:
  $\sqrt{\tfrac{1}{2}}(|1> + |0>)$ and $\sqrt{\tfrac{1}{2}}(|1> + |0>)$
- **Entanglement** simply sums the products:
  $\tfrac{1}{2}(|11> + |10> + |01> + |00>)$

Measurement

- What is **measurement**?
  - Collapse superposition to a single state?
  - Entangle observer with a single state?
- Returns a single state (***value***) ***selected at random*** with probabilities given by $|\psi|^2$
- Superposed value is not accessible after measurement

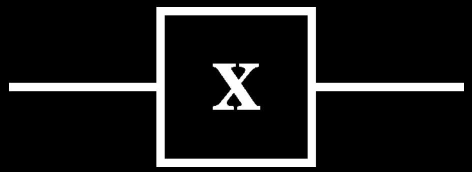
Quantum Processor

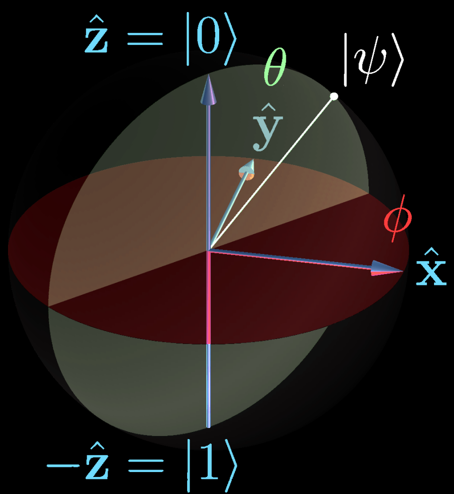
- Gates are not hardware structures
  - Gates operate on qubits “in place”
  - Gates are forces imposed on qubits
  - Conventional computer implements control
- Processor (ideally, one per qubit)
  - Only **thermodynamically reversible** gates
  - No fanout; ancilla

Quantum Gates: Conventional Logic

- Most quantum gates do things you could do with ordinary logic…
- Constraints:
  - Must be reversible
  - Cannot have duplicated inputs
  - Cannot have fanout
    (sequential re-use is OK)

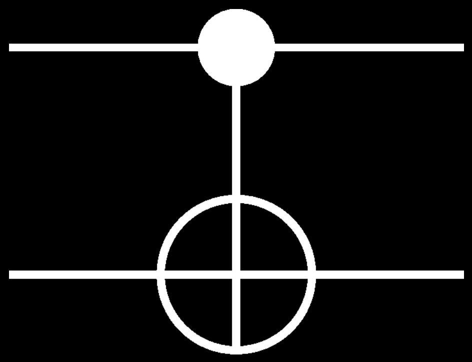
Quantum Gates: Pauli **x**





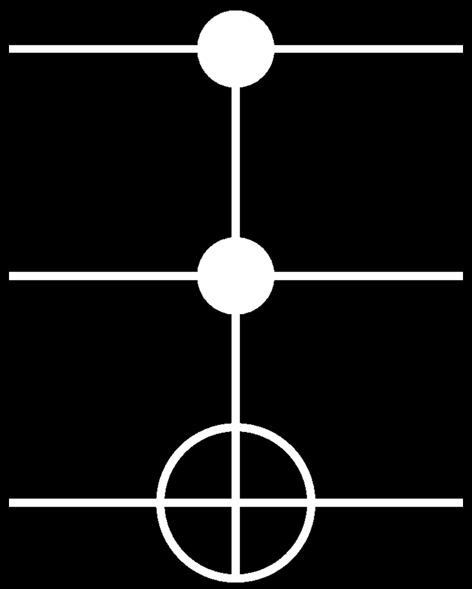
- Pauli **x** is also known as **NOT**
  - Rotates Bloch Sphere around X by π radians
  - Given **|ψ> = a|0> + b|1>**,
    returns **|ψ> = b|0> + a|1>**
  - Functions like conventional **NOT**
  - **x** is its own inverse

Quantum Gates: CNOT



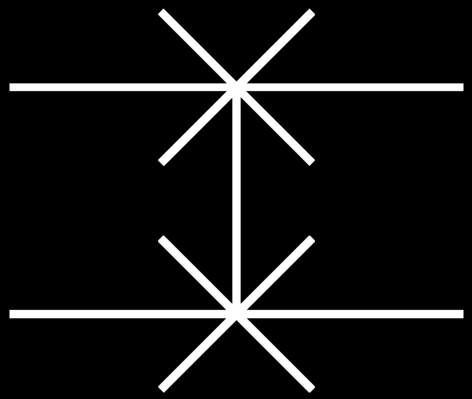
- CNOT is the **Controlled NOT** gate
  - Top input is control, passes thru unchanged
  - Bottom input is inverted where control is 1
  - Both inputs can't be the same qubit
  - Similar to conventional XOR gate

Quantum Gates: CCNOT



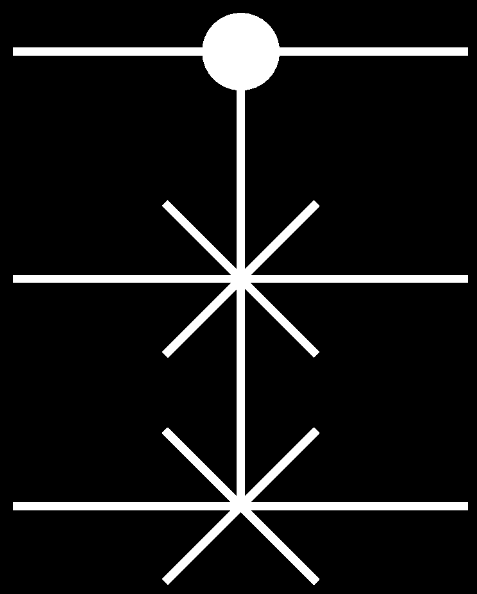
- **CCNOT** is the **Controlled Controlled NOT** gate, also known as **Toffoli** gate
  - A classical universal gate
  - Top two inputs pass unchanged
  - No two inputs can be the same qubit
  - Behaves like `C = (A AND B) XOR C`

Quantum Gates: **`SWAP`**



- **`SWAP`** exchanges the values of two qubits
  - Seems pointless…
    but this is a **reversible assignment**
  - This is also how to *move* qubits to near other qubits so they can interact
  - Both inputs can't be the same qubit

Quantum Gates: CSWAP



- CSWAP is the **Controlled SWAP**, also known as **Fredkin** gate
  - A classical universal gate… and *billiard-ball* conservative
  - Top input passes unchanged
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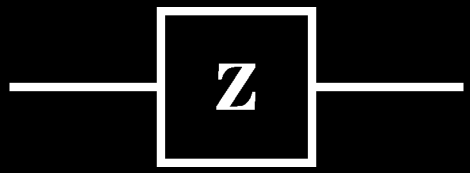
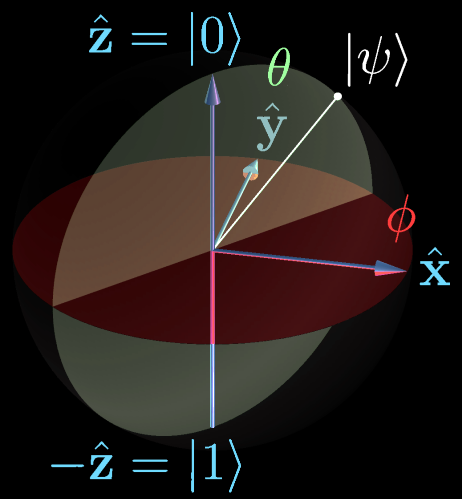
Quantum Gates:
Phase Operations

- Some operations are specific to having two-dimensional (complex) representations, as in PDFs of superposed values
- Precision of phase operations?
  - 180° rotations
  - Arbitrary rotations of unspecified precision are supported in some quantum computers

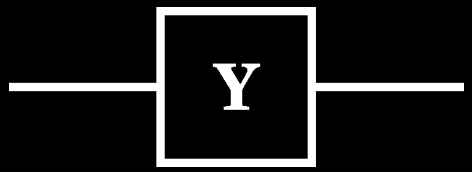
Quantum Gates: Pauli z

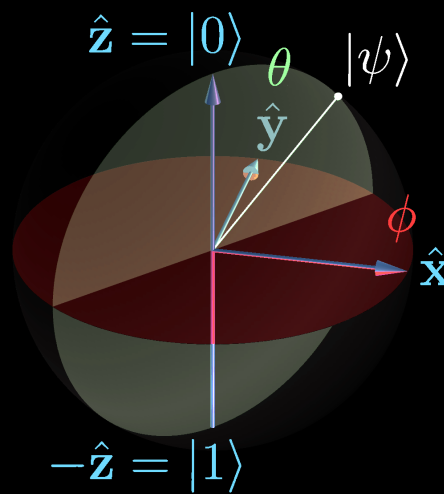




- Pauli z flips phase sign
  - Rotates Bloch Sphere around Z by π radians
  - Given $|\psi\rangle = \sqrt{1/2}(|0\rangle + |1\rangle)$, returns $|\psi\rangle = \sqrt{1/2}(|0\rangle - |1\rangle)$
  - Doesn't alter measured value
  - z is its own inverse

Quantum Gates: Pauli **y**



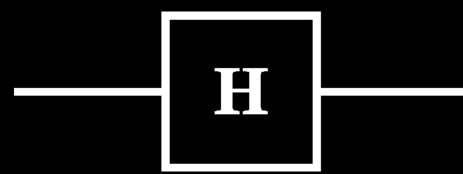


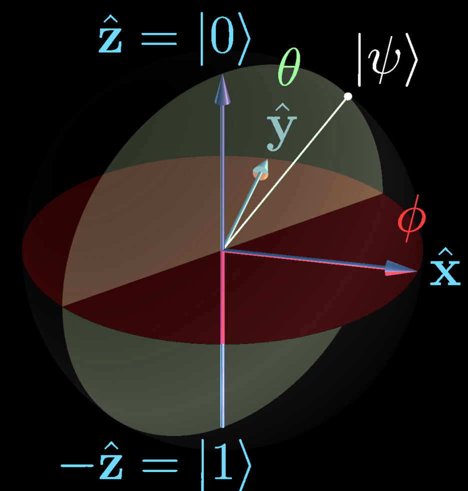
- Pauli **y** like combined **x** and **z**
  - Rotates Bloch Sphere around Y by π radians
  - Given **|ψ> = a|0> + b|1>**,
    returns **|ψ> = ia|1> - ib|0>**
  - **y** is its own inverse

Quantum Gates: Entangled Operations

- Initialization of a qubit is limited:
  - Only conventional, non-superposed, 0 or 1; **no superposed initializers**
  - Often restricted to start of computation
- **All the conventional and phase operations also work on entangled superpositions**

Quantum Gates: Hadamard





- **Hadamard** transforms 0 or 1 into equiprobable superposition
- On Bloch Sphere, like rotate **`y`** by π/2 followed by rotate **`x`** by π
- **`H`** can be an *n*-ary operator, entangling *n* ways
- **`H`** is its own inverse

Quantum Gates: Hadamard

- Equiprobable superpositions created by the Hadamard operator are the primary parallel data structure
- Applying `H` to E qubits is essentially producing 2^E superposed E-bit values from $0..(2^E - 1)$
  - Like initializing v on SIMD PEs with `iproc`
  - Operations on v have parallelism width 2^E

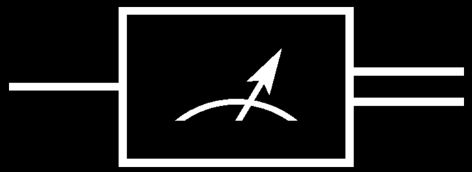
Parallel processing without parallel hardware!

Universal Quantum Gates

- Various options…
  - Rotation in X, Y, Z + Phase + CNOT
  - Clifford set (CNOT, H, S) + T gate
  - Toffoli + H gate
  - Deutsch gate… *which nobody uses*

- Technically, the set of possible quantum gates is uncountable, so any finite sequence of gates only approximates...

Quantum Gates: Measurement



- **Measurement** collapses a superposition
  - Superposed PDF is ***randomly*** sampled
  - Superposed qubit becomes either 0 or 1 (two output lines signify conventional 0/1)
  - PDFs of all entangled qubits prune cases where this qubit was not as measured

Matrix Representations

- A single bit value is $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- A single qubit is: $|a\rangle = v_0|0\rangle + v_1|1\rangle \rightarrow \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$
- The value of a pair of qubits is:

$$|\psi\rangle = v_{00}|00\rangle + v_{01}|01\rangle + v_{10}|10\rangle + v_{11}|11\rangle \rightarrow \begin{bmatrix} v_{00} \\ v_{01} \\ v_{10} \\ v_{11} \end{bmatrix}$$

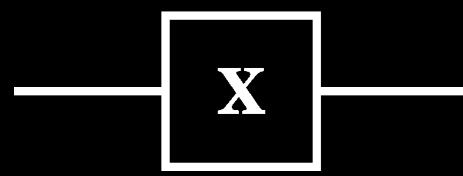
Quantum Operators are Unitary Matrix Operations

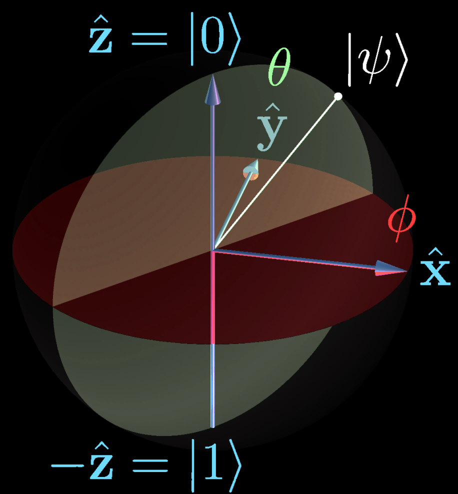
- A unitary matrix is a complex square matrix whose inverse is its conjugate transpose
- Each gate acting on *n* qubits is 2^n x 2^n
- Gate A followed by B is the same as gate B·A
- Tensor/Kronecker product for gates in parallel
- Any unitary can be converted to a set of gates each acting on either one or two qubits

- Measurement is not a unitary operation, and it is not reversible

Quantum Gates: Pauli **x**

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$





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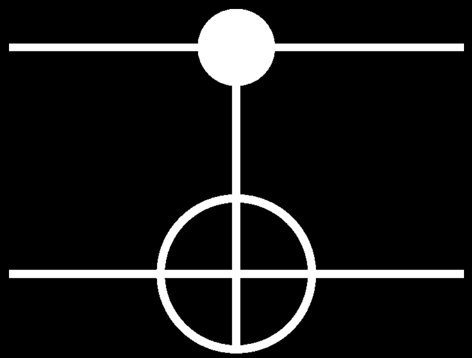
Quantum Gates: **CNOT**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



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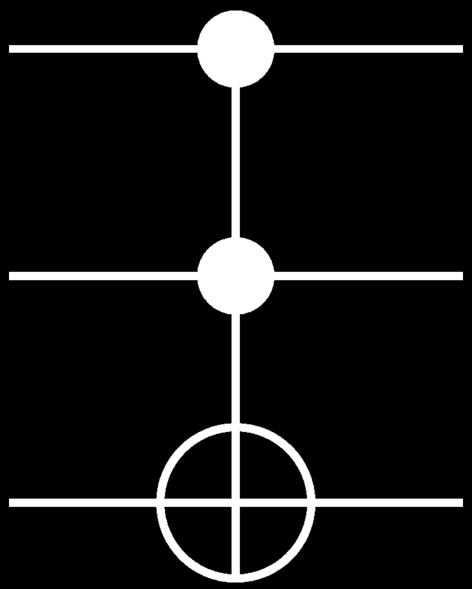
Quantum Gates: CCNOT

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



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Quantum Gates: **`SWAP`**

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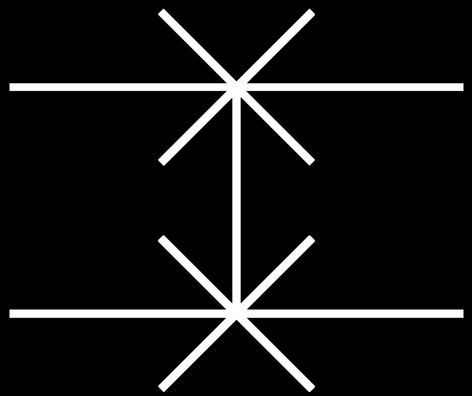

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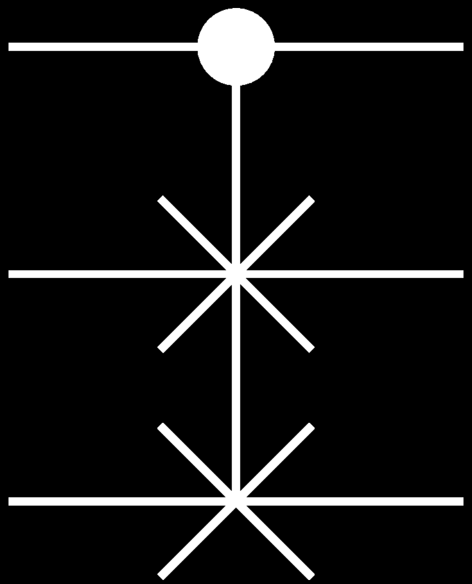
Quantum Gates: CSWAP

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



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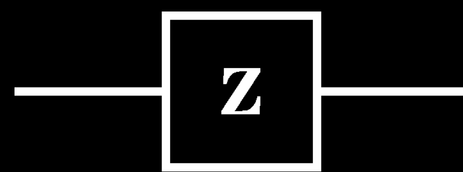
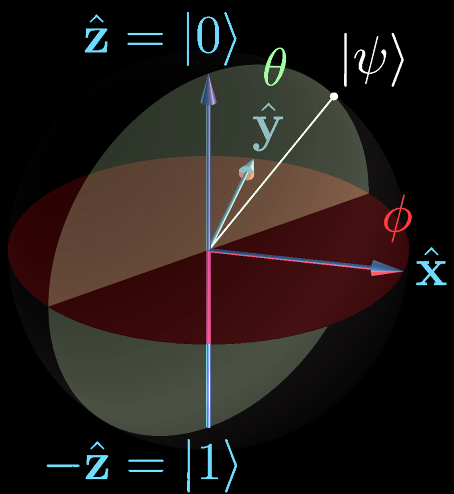
Quantum Gates: Pauli z

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

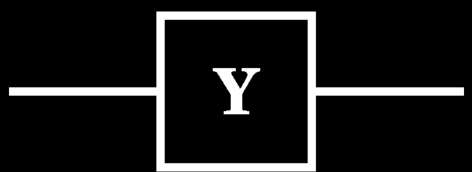


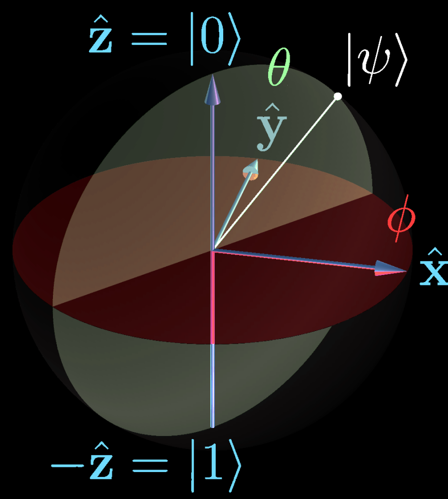


- Pauli **z** flips phase sign
  - Rotates Bloch Sphere around Z by π radians
  - Given **|ψ> = √½(|0> + |1>)**, returns **|ψ> = √½(|0> - |1>)**
  - Doesn't alter measured value
  - **z** is its own inverse

Quantum Gates: Pauli **y**

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$





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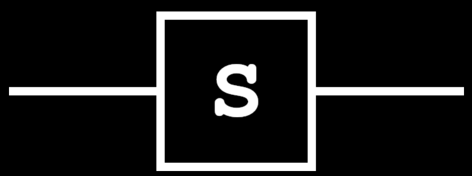
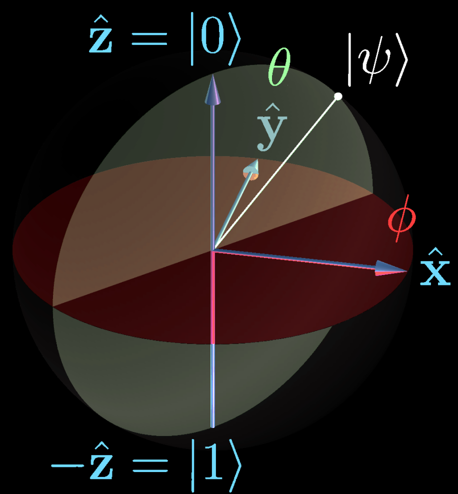
Quantum Gates: Phase

$$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$$

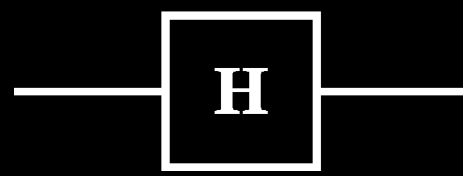




- Phase shift **`S`** by *i*
- There is also a **`P(φ)`** gate that shifts by $e^{i\varphi}$ instead of *i*
- **`P(π/4)`** is also known as **`T`**
- There are parametric rotations about X, Y, Z
- The accuracy of **`φ`**, etc. are unspecified…

Quantum Gates: **Hadamard**

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$



- Two-qubit **Hadamard** is:

$$H_2 = H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$$

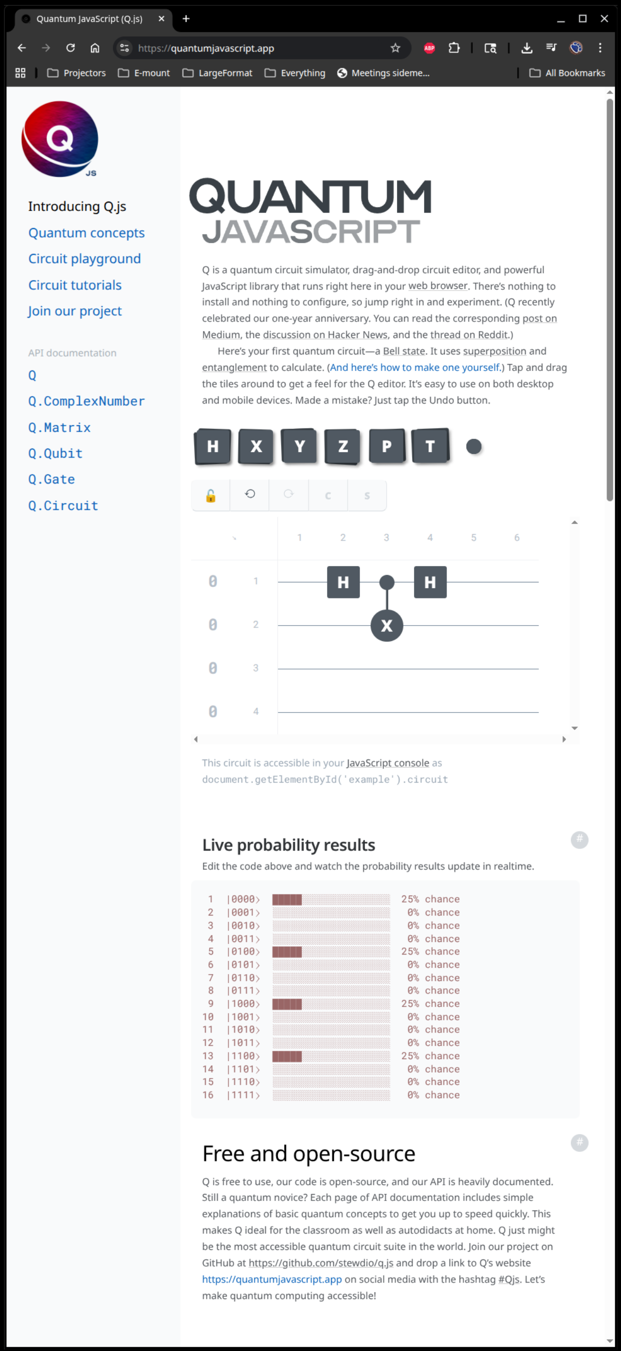
- Applying to two 0 qubits:

$$H_2|00\rangle = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix} = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Quantum Javascript

- **Q.js** drag-and-drop simulator

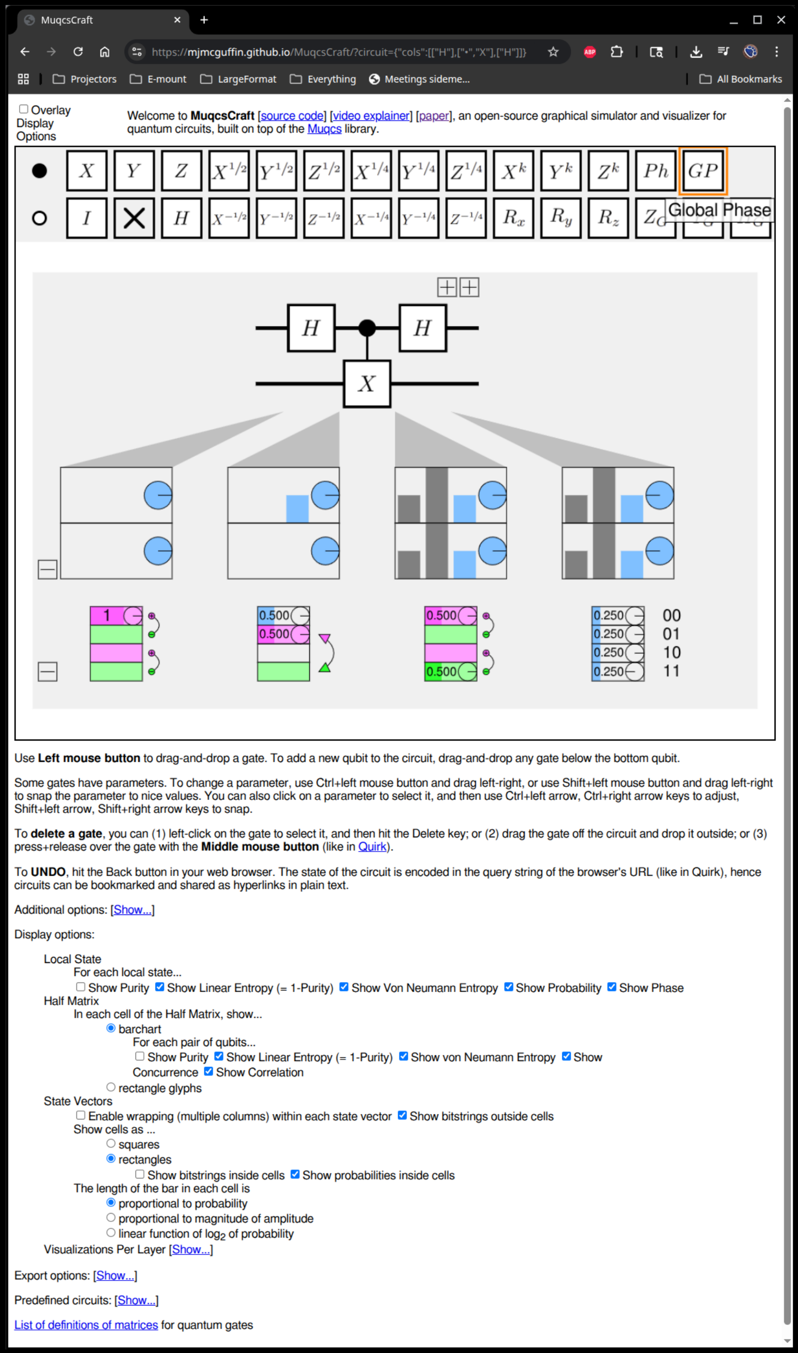
`https://quantumjavascript.app/playground.html`



MuqcsCraft

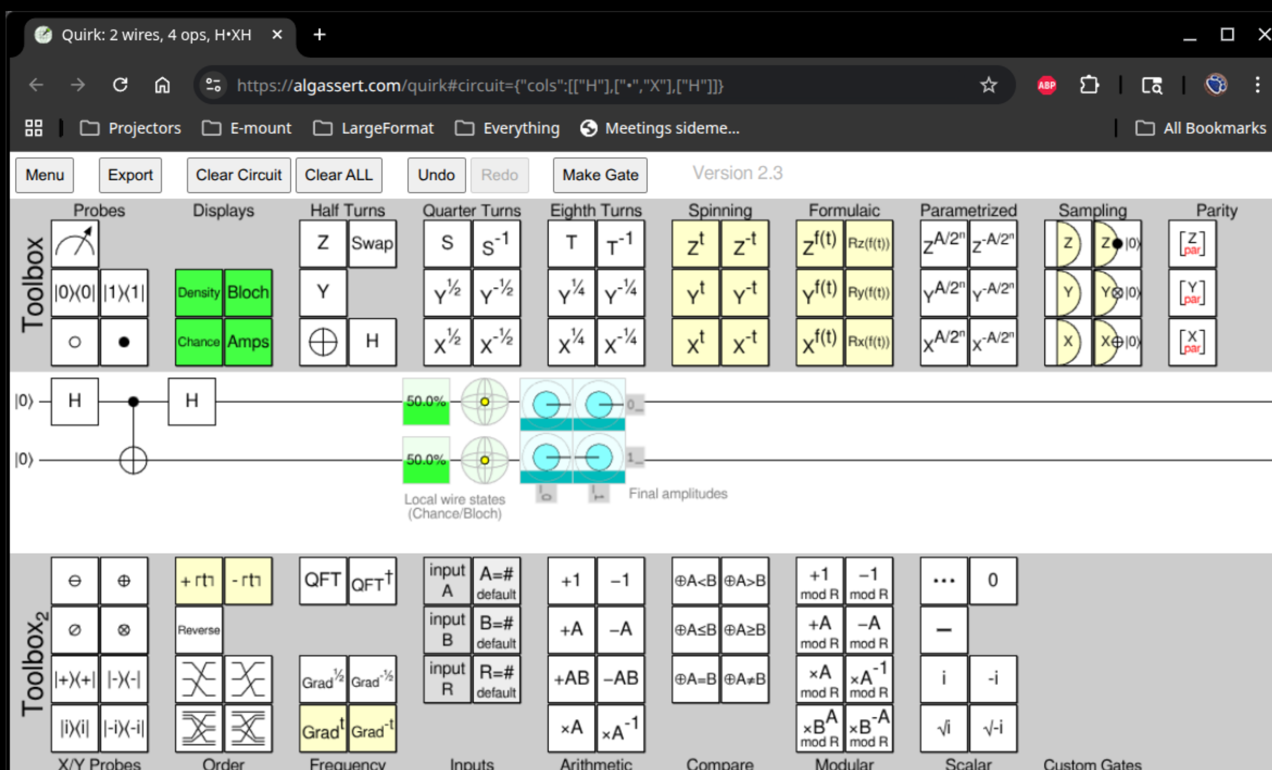
- **MuqcsCraft**

  `https://mjmcguffin.github.io/MuqcsCraft/`



Quirk Circuit Simulator

- **Quirk** open source drag-and-drop simulator
  `https://algassert.com/quirk`



THE QUANTUM LÄND

- **Rydberg Quantum Computers & Simulators**, Stuttgart `https://thequantumlaend.de/quantum-circuit-designer/`