Shor's Algorithm

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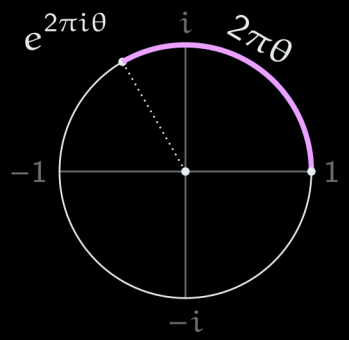
Shor's Algorithm

- The quantum algorithm everyone hears about
  - Fastest algorithm to find prime factors
  - Modern encryption is based on the idea that, given the product of two large primes, it is very difficult to find the two primes
- Shor's algorithm has quantum subroutines, but it is mostly run on a conventional computer
- Not a trivial algorithm...

It's About Phase Estimation

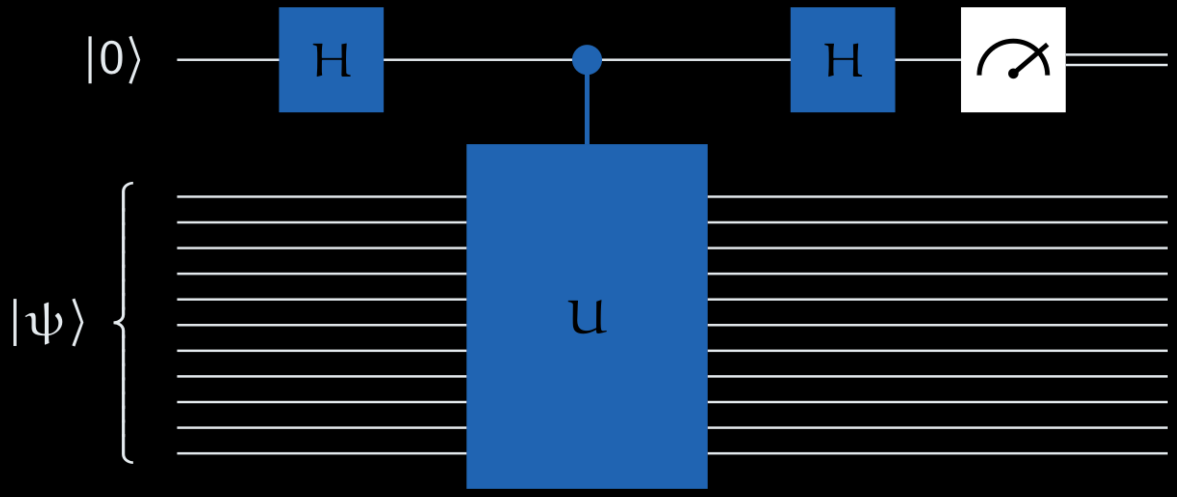


- Given:
  - A unitary quantum circuit U on *n* qubits
  - An n-qubit quantum state $|\psi\rangle$
  - **Promise** that $|\psi\rangle$ is an *eigenvector* of U
- Find an ***m*-bit fraction** approximating $\theta \in [0, 1)$ such that $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$

  Remember that $e^{2\pi i\theta} = \cos(2\pi\theta) + i \sin(2\pi\theta)$

Controlled-U Phase Kickback

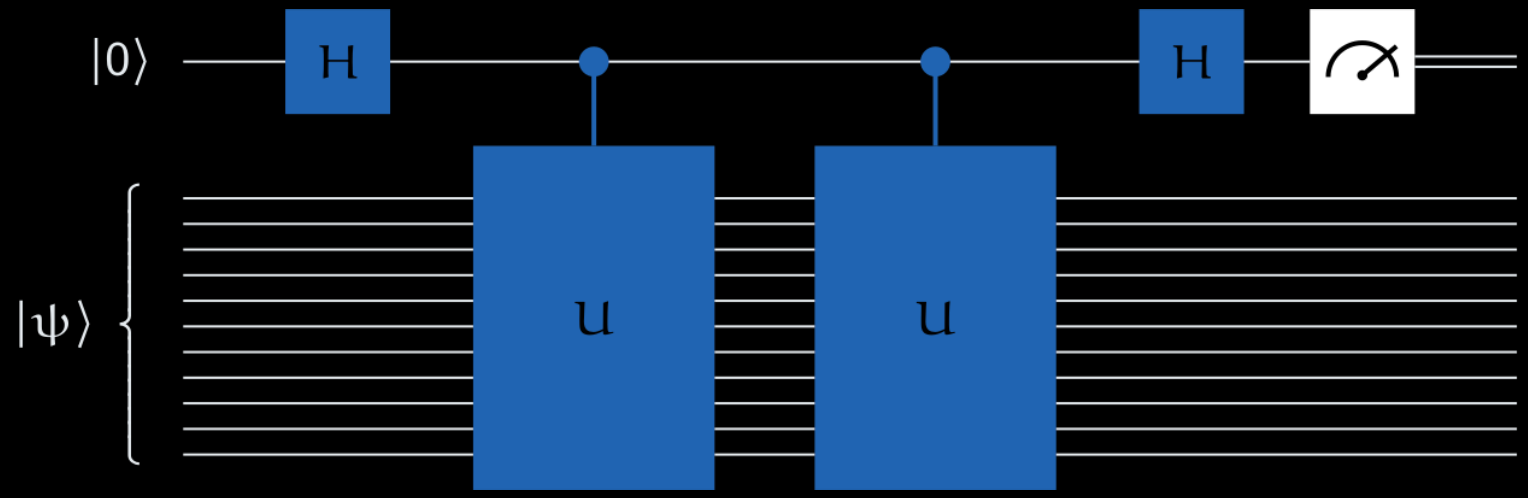
- To apply phase kickback
  - Create a controlled version of U
  - Sample it using this circuit:



- Result is $p(0)=\cos^2(\pi\theta)$ and $p(1)=\sin^2(\pi\theta)$

Phase Kickback Frequency

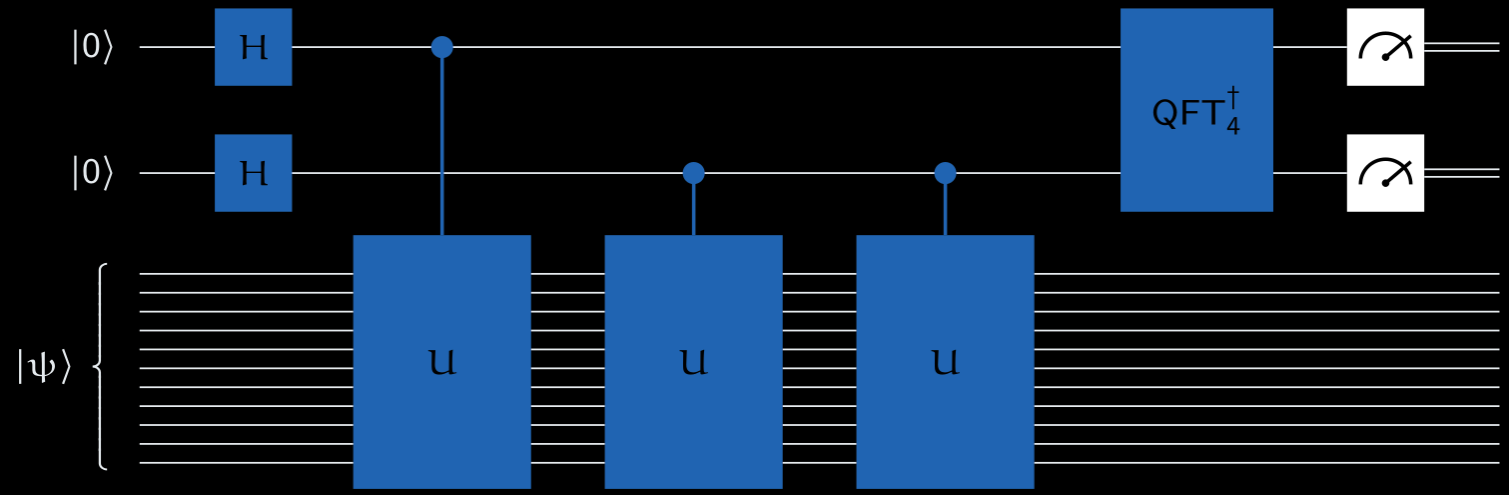
- Applying controlled-U twice
  - Squares the eigenvalue, doubles frequency
  - Sample it using this circuit:



- Want to decompose by both together...

2-bit Phase Estimation

- Composes the single and double applications
  - 2 qubits of phase output to sample
  - Distinguishes 2^2 phases



- What is that QFT_4 block?

Quantum Fourier Transform

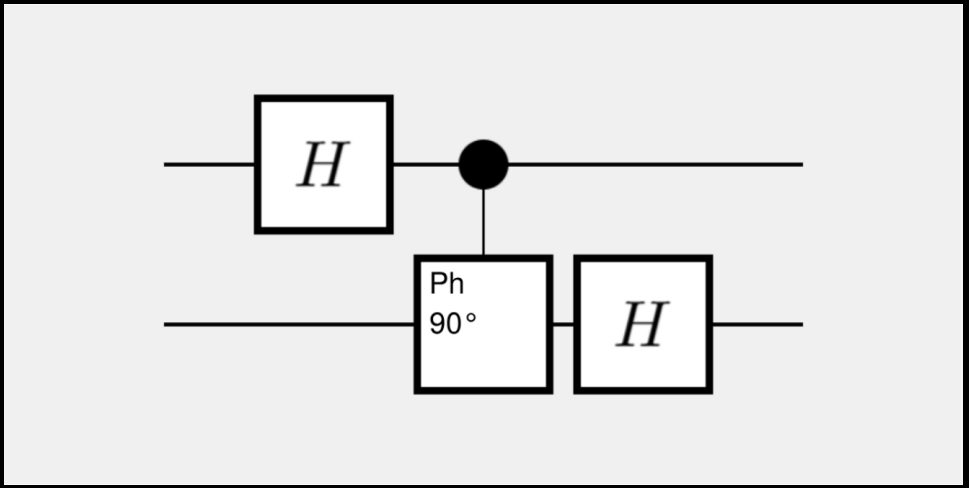
- QFT_4 is a unitary matrix applying the discrete Fourier transform distinguishing 4 phases

$$QFT_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

- What gates implement that?

Quantum Fourier Transform

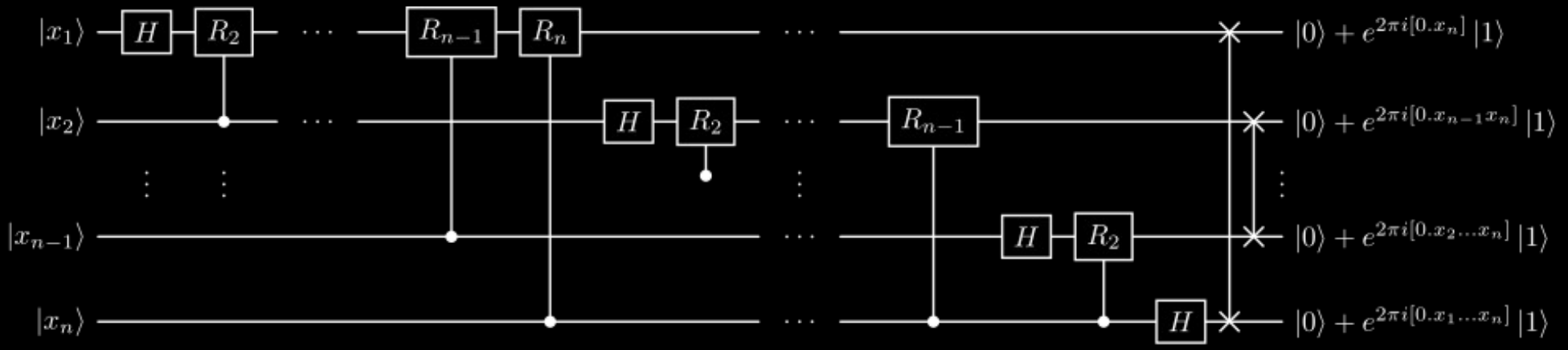
- QFT_4 can be built from H and controlled phase; phase gate is sometimes called R_k for $360/2^k$ º



- Larger QFT have sequences of phase shifts by correspondingly smaller angles

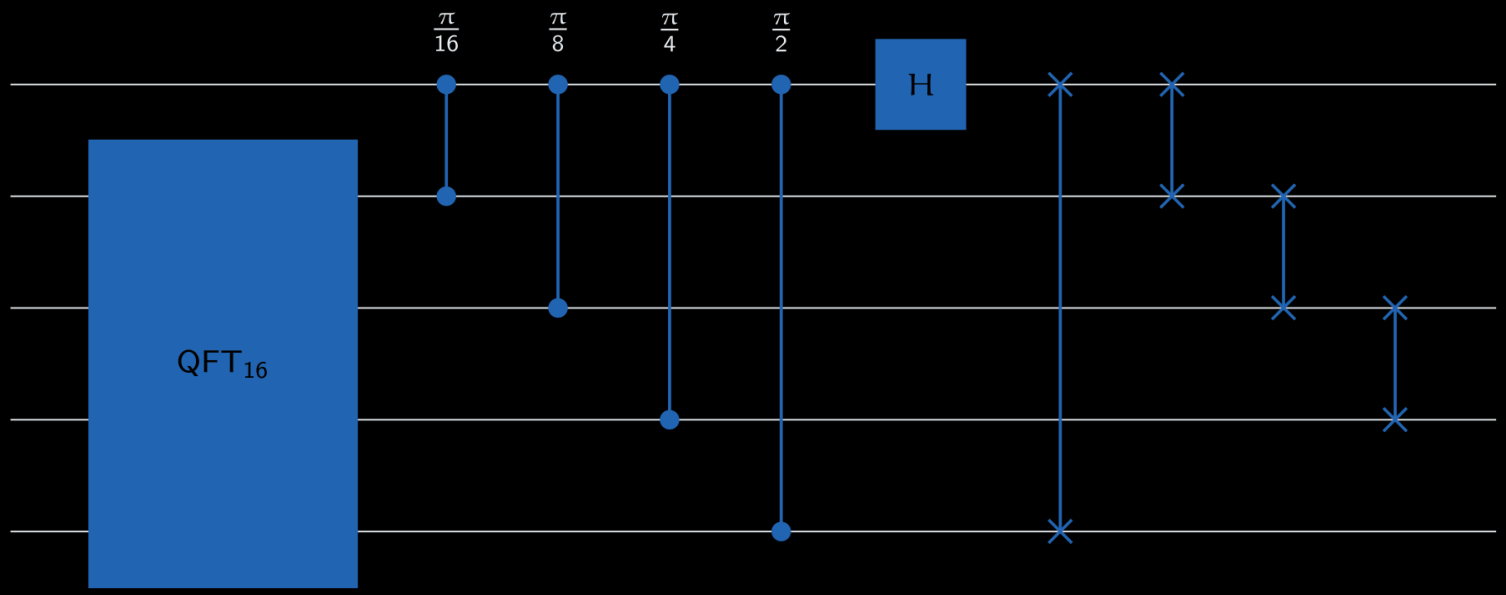
Quantum Fourier Transform

- QFT over *n* qubits is:



Quantum Fourier Transform

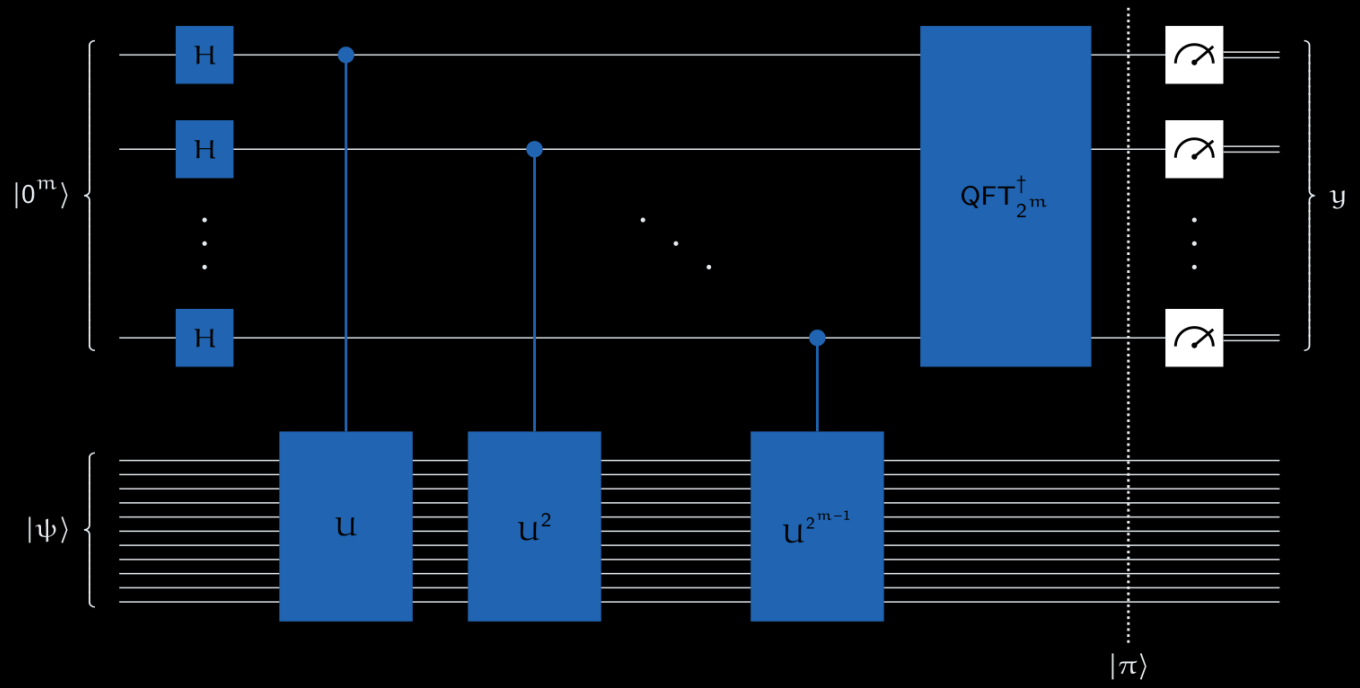
- IBM gives a recursive definition of QFT_{32}:



- Optimizations & approximations can be applied

Phase Estimation

- Evaluation of powers of U gets expensive for estimation with many bits of precision



Phase Estimation

- Probability of measuring the best approximation for y is $\geq 4/\pi^2 \approx 40.5\%$
- Wrong values on either side have lower probabilities
- Can filter by mode of multiple measurements...

Order-Finding Problem

- For each positive integer N,
  define set $\mathbb{Z}_N = \{0,1,\ldots,N-1\}$
  and arithmetic on $\mathbb{Z}_N$ mod N
- Elements $a \in \mathbb{Z}_N$ where gcd(a,N)=1 is $\mathbb{Z}^*_N$

  $\mathbb{Z}^*_{21} = \{1,2,4,5,8,10,11,13,16,17,19,20\}$

- For $a \in \mathbb{Z}^*_N$ $\exists$ positive integer k: $a^k=1$
  … the smallest such k is the order of a in $\mathbb{Z}^*_N$

Order-Finding Example

- For N=21:

  $\mathbb{Z}^*_{21} = \{1,2,4,5,8,10,11,13,16,17,19,20\}$

  $1^1=1$ $2^6=1$ $4^3=1$ $5^6=1$ $8^2=1$ $10^6=1$
  $11^6=1$ $13^2=1$ $16^3=1$ $17^6=1$ $19^6=1$ $20^2=1$

- No efficient classical algorithm is known… solving this would imply an efficient algorithm for integer factorization

Order-Finding by Phase Est.

- Two positive integers a, b are **relatively prime** *iff* they have no common factors: gcd(a,b)=1
- Relative primes are often used for rehashing when there is a collision in a hash function with b buckets:

  If Hash(v) collides, then try (Hash(v)+a)%b, (Hash(v)+a+a)%b, (Hash(v)+a+a+a)%b, …

  All buckets of b will be tried before a repeat...

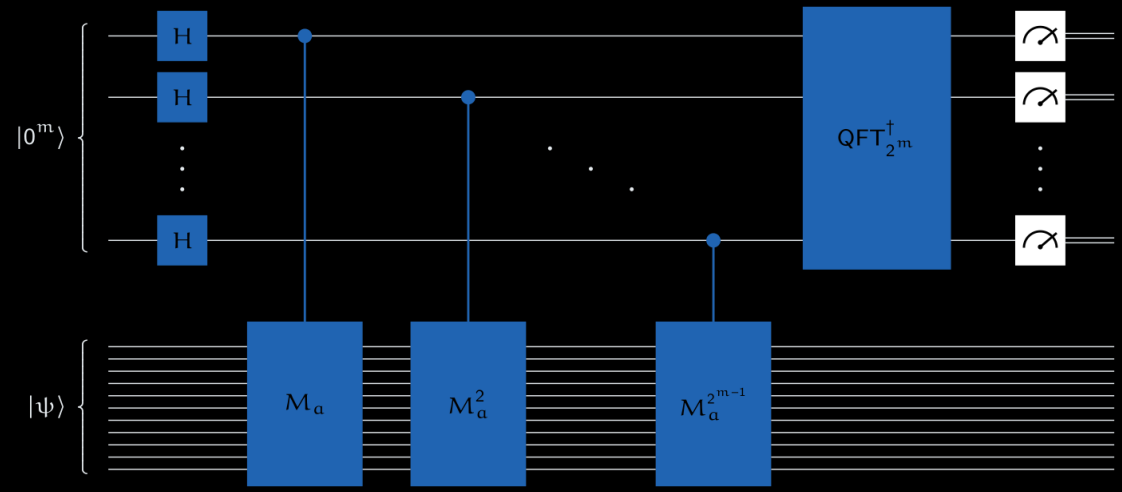
Order-Finding by Phase Est.

- For a given $a \in \mathbb{Z}^*_N$
  $M_a|x\rangle = |ax\rangle$ (for each $x \in \mathbb{Z}_N$)
- For N=8 and a=3:

  $M_3|0\rangle=|0\rangle$ $M_3|1\rangle=|3\rangle$ $M_3|2\rangle=|6\rangle$ $M_3|3\rangle=|1\rangle$
  $M_3|4\rangle=|4\rangle$ $M_3|5\rangle=|7\rangle$ $M_3|6\rangle=|2\rangle$ $M_3|7\rangle=|5\rangle$
- Because 3,8 are **relatively prime**, $M_a|x\rangle$ is a **permutation**, so it is a **unitary matrix!**

Order-Finding by Phase Est.

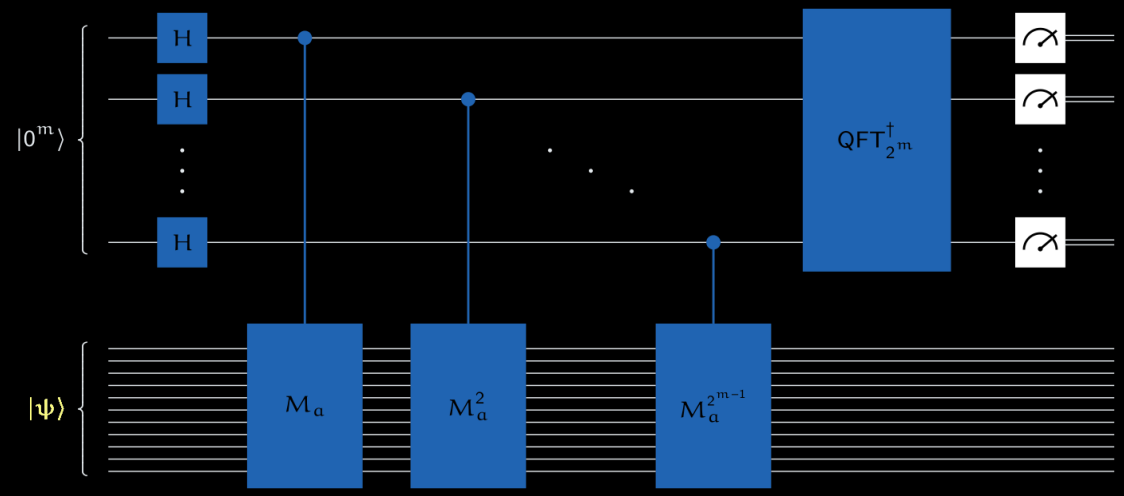
- Find order of $a \in \mathbb{Z}^*_N$ by estimating phase of M_a



- Need $M_a{}^k$ for k is 2^0, 2^1, 2^2, … 2^{m-1}
  where $M_a{}^k$ is a^k % N, computable with $O(n^2)$ cost

Order-Finding by Phase Est.

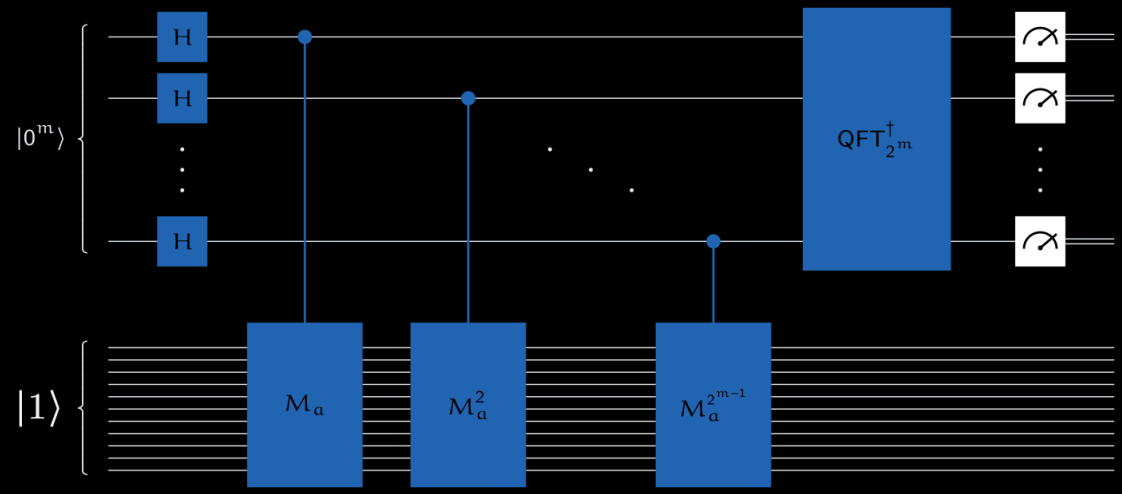
- Find order of $a \in \mathbb{Z}^*_N$ by estimating phase of M_a



- But how do I find an eigenvector $|\psi\rangle$?
  We can simply use $|1\rangle$

Order-Finding by Phase Est.

- Total cost is **O(n^3)** for *n*-bit values



- m Hadamard gates, cost O(*n*)
- m controlled unitary operations, cost **O(n^3)**
- one Quantum Fourier transform, cost O(n^2)

Algorithm Factoring N

1. Pick random $a \in \{2, \ldots, N-1\}$
2. Let d = **gcd(a,N)**
3. If $d \geq 2$, d is a factor and we're done
4. **Let r = order of a % N**
5. If r is odd, fail
6. Let d = **$\gcd(a^{r/2} - 1, N)$**
7. If $d \geq 2$, d is a factor; else fail

Greatest Common Divisor

- Euclidean algorithm for gcd(a,N) given N ≥ a

```
int
gcd(int a, int N)
{
  return(a ? gcd(N % a, a) : N);
}
```

- Quite fast on conventional hardware
- If gcd(a,N)≡1 then a, N are relatively prime

Conventional Order Finding

- Sequential order(a,N) is O(N), i.e., $O(2^n)$

```
int order(int a, int N) {
  int p=0;
  for (int i=0; i<N; ++i) {
    if (p == 1) return(i);
    p += a;
    if (p >= N) p -= N;
  }
  return(1); // failure
}
```

Shor's Algorithm in bc

- Really easy, really slow for big N...

```
# recursive GCD function
define g(a,n) {
  if (a) return(g(n % a, a))
  return(n)
}

# order of a % n
define o(a,n) {
  v=0
  for (i=0; i<n; ++i) {
    if (v==1) return(i)
    v += a
    if (v >= n) v -= n
  }
  return(-1) # failed to find order
}

# power e of a mod n
define p(a,e,n) {
  q=1
  while (e > 0) {
    if (e % 2) q = (a * q) % n
    a = (a * a) % n
    e = e / 2
  }
  return(q)
}
```

```
# factor n using guess a
define f(n,a) {
  d = g(a,n)
  if (d > 1) return(d)
  r = o(a,n)
  if (r % 2) return(1)
  w = p(a,r/2,n)
  if (w < 2) return(1)
  d = g(w-1,n)
  if (d > 1) return(d)
  return(1)
}

f(1679, 132)
1

f(1679, 318)
23

f(1679, 1197)
23

1679/23
73
23*73
1679
```

Algorithm Factoring N

- How often does this fail?
  Claimed less than 50% of the time!
- Very unlikely we're done in step 3, but ...
  - $a^r \% N \equiv 1$ *iff* $(a^r - 1) \% N \equiv 0$
  - If r is even, $(a^r - 1) = (a^{r/2} + 1)(a^{r/2} - 1)$ and a prime factor of N is likely to divide $(a^{r/2} - 1)$
  - Factoring values of N that have just two prime factors, it seems to fail more often?

Shor's Algorithm in `qiskit`

- Here's factoring 15 using `qiskit`:
  `https://quantum.cloud.ibm.com/docs/en/tutorials/shors-algorithm`